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A Probabilistic Approach to the Models of Spin Glasses

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Introducing the notions of quenched and annealed probability measures, a systematic study of some problems in the description of spin glasses is attempted. Inequalities and variational principles for the free energies are derived. The absence of spontaneous breakdown of the gauge symmetry is discussed and some high-temperature properties are studied. Examples of annealed models with more than one phase transition are shown.

KEY WORDS: Ising; spin glass.

1. INTRODUCTION

A commonly accepted way to describe disordered systems is to represent them by statistical ensembles in which the degrees of freedom are coupled by random interactions. This additional randomness may be treated in different ways: Random interactions may be considered as new degrees of freedom, and in extreme cases they may be in thermal equilibrium with the rest of the system (annealed state) or completely frozen in some random position (quenched state). Many years after the pioneering work of Brout[(1)] the quenched state of certain spin models became the center of interest of the theoretical research on spin glasses. Edwards and Anderson[(2)] pointed out that these systems can properly be described by the quenched state of randomly interacting spin models on regular lattices. Meanwhile, one encounters two main difficulties: the first is to calculate, in a respectable approximation, the quenched free energy, and the second is to give a reliable proof that there exists a phase transition—in the static sense—between the high-temperature paramagnetic and the low-temperature spin

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glass state. Neither of these problems has got so far a reassuring solution, excepted probably in the case of the so-called Sherrington–Kirkpatrick model.[3]

While not pretending to contribute to the solution of these great questions, in the present paper we attempt a systematic study of what were called the "annealed" and "quenched" states. To this end we define in Section 2 the annealed and quenched probability measures which play the same role as the Gibbs measure in equilibrium systems. In Section 3 we derive some inequalities for the quenced free energy, and in Section 4 we establish a variational principle characterizing both the annealed and quenched free energies in the space of the joint probability distributions for spins and bonds. In Section 5 we raise the question of the uniqueness of the quenched state and show that the gauge invariance cannot be broken by different choices for the boundary condition. This suggests that the "order parameter" of the spin glass state must be the expectation value of a gauge-invariant and nonlocal observable. The order parameter, proposed by Edwards and Anderson,[2] has indeed these properties, as we point out in Section 6. A discussion of high-temperature properties is also given there and the functional relationship between order parameter and free energy is established. Finally in Section 7 we return to the study of annealed models and give examples for one, two, and three consecutive phase transitions in such models.

2. DEFINITION OF THE SYSTEM

Let L be a lattice; at each site i of L is associated a single spin space S, where S is a subset of $\mathbb{R}^\nu$. The spin configurations are defined by $\sigma: L \to S$ and the formal Hamiltonian of the system is given as

$$H(J, \sigma) = - \sum_{b \subset L} J_b \phi_b(\sigma) \tag{2.1}$$

Here the ϕ_b's are bounded, real-valued functions depending on $\sigma_b = \{\sigma_i; i \in b\}$ and the J_b's are real random variables with probability distribution $d\rho_b$ the mean value of which is finite. The finite partial sums of (2.1) are well defined for any σ with ρ probability 1. In particular,

$$H_V(J, \sigma) = - \sum_{b \subset V} J_b \phi_b(\sigma) \tag{2.2}$$

exists for all finite $V \subset L$.

On S is given an *a priori* finite measure $d\mu_0$ not necessarily normalized and $d\mu_V(\sigma)$ denotes $\prod_{i \in V} d\mu_0(\sigma_i)$, while $d\rho_V(J) = \prod_{b \subset V} d\rho_b(J_b)$. In the following discussion we consider only finite volumes and we omit the label V.

For given interactions J, the *free energy* $F(J)$ and the *equilibrium state* at inverse temperature β are defined as usual by

$$e^{-\beta F(J)} = \int d\mu(\sigma)\, e^{-\beta H(J,\sigma)} \tag{2.3}$$

and by the Gibbs probability measure

$$dG_J(\sigma) = g(J,\sigma)\, d\mu(\sigma)$$

where

$$g(J,\sigma) = e^{\beta[F(J) - H(J,\sigma)]} \tag{2.4}$$

(2.4) can also be written as

$$F(J) = H(J,\sigma) + \frac{1}{\beta} \ln g(\bar{J},\sigma) \tag{2.5}$$

which yields (since the Gibbs measure is normalized)

$$\begin{aligned} F(J) &= \int d\mu(\sigma)\, g(J,\sigma) \left[H(J,\sigma) + \frac{1}{\beta} \ln g(J,\sigma) \right] \\ &= E(J) - \frac{1}{\beta} S(J) \end{aligned} \tag{2.6}$$

The *quenched free energy* $\bar{F}$ and the *quenched state* are defined by

$$\bar{F} = \int d\rho(J)\, F(J) = -\frac{1}{\beta} \int d\rho(J) \ln\left[\int d\mu(\sigma)\, e^{-\beta H(J,\sigma)} \right] \tag{2.7}$$

and by the quenched probability measure

$$dQ(J,\sigma) = g(J,\sigma)\, d\mu(\sigma)\, d\rho(J) \tag{2.8}$$

which yields

$$\bar{F} = \int dQ(J,\sigma) \left[H(J,\sigma) + \frac{1}{\beta} \ln g(J,\sigma) \right] = \bar{E} - \frac{1}{\beta} \bar{S} \tag{2.9}$$

We shall impose that the distribution $d\rho_b(x)$ falls off sufficiently rapidly so that $\bar{F}$ is well defined.

The *annealed free energy* F_{an} and the *annealed state* are defined by the prescription that the average over the interactions has to be performed in the partition function, i.e.,

$$F_{\text{an}} = -\frac{1}{\beta} \ln\left[\int d\rho(J)\, e^{-\beta F(J)} \right] = -\frac{1}{\beta} \ln\left[\int d\rho(J)\, d\mu(\sigma)\, e^{-\beta H(J,\sigma)} \right] \tag{2.10}$$

and by the annealed probability measure

$$dA(J,\sigma) = h(J,\sigma)\, d\mu(\sigma)\, d\rho(J)$$

where

$$h(J,\sigma) = e^{\beta[F_{\mathrm{an}} - H(J,\sigma)]} \tag{2.11}$$

i.e.,

$$F_{\mathrm{an}} = H(J,\sigma) + \frac{1}{\beta}\ln h(J,\sigma) \tag{2.12}$$

which yields

$$F_{\mathrm{an}} = \int dA(J,\sigma)\left[H(J,\sigma) + \frac{1}{\beta}\ln h(J,\sigma)\right] = E_{\mathrm{an}} - \frac{1}{\beta}S_{\mathrm{an}} \tag{2.13}$$

Introducing the space $\mathscr{B}$ of joint probability distributions

$$\mathscr{B} = \left\{ f = f(J,\sigma); f(J,\sigma) \geqslant 0 \int d\rho\, d\mu\, f = 1 \right\} \tag{2.14}$$

and the *free energy functional* $F[f]$ defined on $\mathscr{B}$ by

$$F[f] = \int d\rho(J)\, d\mu(\sigma)\, f(J,\sigma)\left[H(J,\sigma) + \frac{1}{\beta}\ln f(J,\sigma)\right] \tag{2.15}$$

we can express $\bar{F}$ and F_{an}, respectively, as

$$\bar{F} = F[g], \qquad F_{\mathrm{an}} = F[h] \tag{2.16}$$

Finally, for any subset $B = (b_1, \ldots, b_k)$ of b's in V we introduce $J_B = \{J_b\}_{b\in B}$ and

$$\bar{F}(J_B) = \int \prod_{\substack{b\subset V\\ b\notin B}} d\rho_b(J_b)\, F(J) \tag{2.17}$$

which implies in particular

$$\bar{F}(J_B) = F(J) \qquad \text{if} \quad B = \{b; b\subset V\}$$

$$\bar{F}(J_B) = \bar{F} \qquad \text{if} \quad B = \varnothing$$

Let us note that for any $J'_B = \{J'_b\}_{b\in B}$, $\bar{F}(J'_B)$ represents the quenched free energy with respect to the new measure $d\rho'(J)$ where

$$d\rho'(J) = \prod_{\substack{b\subset V\\ b\notin B}} d\rho_b(J_b) \prod_{b\in B} \delta(J_b - J'_b)\, dJ_b$$

3. INEQUALITIES OF THE QUENCHED FREE ENERGY

In this section we first collect inequalities for $\bar{F}(J_B)$; we then discuss the dependence of $\bar{F}$ on the distribution $d\rho$.

Proposition 3.1.

(i) $$\bar{F}(\bar{J}_B) \leqslant \bar{F}(\bar{J}_{B'}) \qquad \text{for any} \quad B \subset B' \tag{3.1}$$

where $\bar{J}_b = \int d\rho_b(x)\, x$;

(ii)
$$F_{\rm an} \leqslant \bar{F} \leqslant F(\bar{J}) \tag{3.2}$$

The proof of (i) follows from Jensen's inequality using the known fact that $F(J)$ is a concave function of each J_b's and thus $F(J_B)$ is also a concave function of each J_b's, $b \in B$. The proof of (ii) follows from Jensen's inequality using the fact that $F = -\beta \ln Q$ is a convex functional of the partition function Q.

This proposition was earlier published by Rosa[4]; as mentioned in Section 2, the inequalities (3.1) can be regarded as the comparison of two different averages of the same function $F(J)$; we thus have

$$\int d\rho_1 F \leqslant \int d\rho_2 F \tag{3.3}$$

where the probability measure $d\rho_2$ is sharper than $d\rho_1$.

The question naturally arises whether (3.3) is generally true, i.e., whether a "sharpening" of the distribution causes the quenched free energy to increase. The inverse problem is also of interest: does a broadening decrease F? At first we show that the broadening problem can always be solved.

Lemma 3.1. Let $f = \mathbb{R} \to \mathbb{R}$ be concave and let ρ and ν be two probability measures on $\mathbb{R}$ such that

(i) $\int d\rho(x)\, f(x)$ exists

(ii) $\int d\nu(x)\, x = 0$ and

$\int d\nu(x)\, f(x+y)$ exists for all y in $\mathbb{R}$

then

$$\int d(\rho * \nu)(x)\, f(x) \leqslant \int d\rho(y)\, f(y) \tag{3.4}$$

where $d(\rho * \nu)$ denotes the convolution of the measures.

Proof. Using Jensen's inequality together with $\int d\nu(x)\, x = 0$ implies

$$\int d\rho(x)\, f(x) \geqslant \int d\rho(x) \int d\nu(y)\, f(x+y) = \int d(\rho * \nu)(y)\, f(y)$$

This lemma can be immediately extended to concave functions of several variables and implies the following result. ■

Proposition 3.2. Let $\{d\nu_b\}$, $b \subset V$, be a set of probability measures with zero mean and such that $\int \prod_{b \subset V} d\nu_b(J_b)\, F(J_b + \tilde{J}_b)$ exists for all $\tilde{J}$; then

$$\int d(\rho * \nu)(J)\, F(J) \leqslant \int d\rho(J)\, F(J) = \bar{F} \tag{3.5}$$

Remarks. 1. If E denotes the mean value and Δ^2 the mean square deviation then $E_{\rho * \nu} = E_\rho + E_\nu$ and $\Delta^2_{\rho * \nu} = \Delta^2_\rho + \Delta^2_\nu$. Hence in Proposition 3.2 we have $E_{\rho * \nu} = E_\rho$ and $\Delta^2_{\rho * \nu} \geqslant \Delta^2_\rho$, so that $\rho * \nu$ is indeed broadened in comparison to ρ and does decrease $\bar{F}$.

2. Replacing $d\rho_b$ by $d(\rho * \nu_b)$ corresponds to replacing J_b by $J_b + \xi_b$ where ξ_b is a random variable independent of J_b and distributed according to $d\nu_b$.

3. The inequalities (3.1) follow from this theorem if in the latter we replace $\rho_b(x)$ by $\delta(x - \bar{J}_b)$ for $b \in B'$ and take

$$\begin{aligned} \nu_b(x) &= \delta(x) && \text{for} \quad b \notin B' \quad \text{and} \quad b \in B \\ \nu_b(x) &= \rho_b(x + \tilde{J}_b) && \text{for} \quad b \in B'/B \end{aligned}$$

In view of the second remark, the sharpening problem can be formulated in this way: given a random variable J_b we have to find a nontrivial decomposition of J_b into the sum of two independent random variables, one of them having zero mean. This problem cannot be generally solved. However, if the J_b's are Gaussian random variables, then such a decomposition is possible[5] and we have the following result.

Proposition 3.3. Consider the quenched free energy associated with the two different set of Gaussian distributions $\{\rho_b^{(1)}\}$ and $\{\rho_b^{(2)}\}$ such that $E_{\rho_b^{(1)}} = E_{\rho_b^{(2)}}$ and $\Delta^2_{\rho_b^{(1)}} \geqslant \Delta^2_{\rho_b^{(2)}}$ for all b. Then

$$\int d\rho^{(1)}(J)\, F(J) \leqslant \int d\rho^{(2)}(J)\, F(J) \tag{3.6}$$

Proof. Let ν_b be the Gaussian measure with zero mean and mean square deviation $\Delta^2_{\rho_b^{(1)}} - \Delta^2_{\rho_b^{(2)}}$. Then $\rho_b^{(1)} = \rho_b^{(2)} * \nu_b$ and the statement follows from Proposition 3.2. ■

There is another way to generalize the inequality (3.1) which shows that the approximation of a given distribution with a suitably chosen discrete distribution results in the increase of the quenched free energy. The better the discrete distribution approaches the original one the smaller is the upper bound.

Proposition 3.4. Let $d\rho_b(x) = d\rho_b(-x)$ for all b and let

$$d\nu_b(x) = \tfrac{1}{2}\left[\delta(x - |\overline{J_b}|) + \delta(x + |\overline{J_b}|)\right] \qquad \text{where} \quad |\overline{J_b}| = \int d\rho_b(x)\,|x|$$

then

$$\bar{F} \leqslant \int d\nu(J)\, F(J) \leqslant F(\bar{J})$$

Proof. We may fix the interactions with the exception of the single J_b and it is sufficient to show that for the concave function $F(J_b)$ the inequalities

$$\int_{-\infty}^{\infty} F(x)\, d\rho_b(x) \leqslant \tfrac{1}{2}\left[F(|\overline{J_b}|) + F(-|\overline{J_b}|)\right] \leqslant F(0)$$

hold. Here the second inequality comes from the definition of a concave function. Now $2d\rho_b$ is a probability measure on both $\mathbb{R}^+$ and $\mathbb{R}^-$ and hence

$$2\int_{-\infty}^{0} F(x)\, d\rho_b(x) \leqslant F\left[2\int_{-\infty}^{0} x\, d\rho_b(x)\right]$$

$$2\int_{0}^{\infty} F(x)\, d\rho_b(x) \leqslant F\left[2\int_{0}^{\infty} x\, d\rho_b(x)\right]$$

But

$$2\int_{0}^{\infty} x\, d\rho_b(x) = -2\int_{-\infty}^{0} x\, d\rho_b(x) = \int_{-\infty}^{\infty} |x|\, d\rho_b(x)$$

which concludes the proof. ■

The generalization of this proposition for noneven distributions and for more detailed partitions of the support of ρ is easy and we leave it to the reader.

4. VARIATIONAL PRINCIPLES

The existence of a variational principle for the free energy is a manifestation of equilibrium. Since the quenched system is not in equilibrium we cannot expect that a variational principle completely analogous to that of equilibrium systems will also hold for $\overline{F}$. In mean field calculation (Edwards and Anderson,[2] Sherrington and Kirkpatrick[3]) the free energy of the quenched state at low temperature is above the continuation from high temperature, this continuation being essentially the annealed free energy (see Proposition 3.1). In this section we give some insight on this mean field result by showing that the quenched free energy satisfies a variational principle on a subspace $\mathscr{B}_0$ of the space $\mathscr{B}$ (2.14); on the other hand, the annealed free energy satisfies a variational principle on the whole space $\mathscr{B}$.

Let us recall that for given interaction J, the variational principle for the finite volume free energy is expressed by the inequality

$$F(J) = F[J;\, g_J] \leqslant F[J;\, f] \tag{4.1}$$

where the free energy functional is defined by

$$F[J;\, f] = \int d\mu(\sigma)\, f(\sigma)\left[H(J,\sigma) + \frac{1}{\beta}\ln f(\sigma)\right] \tag{4.2}$$

on the space of distribution functions $f = f(\sigma)$, and $g_J = g(J,\sigma)$ is the Gibbs function (2.4).

Indeed the inequality (4.1) is a consequence of the inequality

$$F[J;f] - F[J] = -\frac{1}{\beta}\int d\mu(\sigma)\, f(\sigma) \ln \frac{g(J,\sigma)}{f(\sigma)} \geqslant 0$$

For random systems we consider the free energy functional (2.15) defined on $\mathscr{B}$ and we also introduce $\mathscr{B}_0$ the subspace of $\mathscr{B}$ defined by

$$\mathscr{B}_0 = \left\{ f = f(J,\sigma);\, f(J,\sigma) \geqslant 0,\, \int d\mu(\sigma)\, f(J,\sigma) = 1,\, \forall J \right\} \tag{4.3}$$

Proposition 4.1.

(1)
$$F_{\text{an}} = F[h] = \min_{f \in \mathscr{B}} F[f] \tag{4.4}$$

(2) For any $f \in \mathscr{B}$

$$F[f] - \int d\rho(J)\, d\mu(\sigma)\, f(J,\sigma) F(J) \geqslant 0 \tag{4.5}$$

and the equality holds for $f = g(J,\sigma)$

(3)
$$\bar{F} = F[g] = \min_{f \in \mathscr{B}_0} F[f] \tag{4.6}$$

Proof. (1) Using the definitions (2.12), (2.13), (2.15), we have

$$F[f] - F_{\text{an}} = -\frac{1}{\beta}\int d\rho(J)\, d\mu(\sigma)\, f(J,\sigma) \ln \frac{h(J,\sigma)}{f(J,\sigma)} \geqslant 0$$

(2)
$$F[f] - \int d\rho(J)\, d\mu(\sigma)\, f(J,\sigma) F(J)$$

$$= -\frac{1}{\beta}\int d\rho(J)\, d\mu(\sigma)\, f(J,\sigma) \ln \frac{g(J,\sigma)}{f(J,\sigma)} \geqslant 0$$

(3) Using (4.5) we have for any $f \in \mathscr{B}_0$

$$F[f] \geqslant \bar{F}$$

which concludes the proof. ■

Another way to formulate the above result is the following: In the space $\mathscr{B}$ of the joint probability distributions the minimization of $F[f]$ yields the annealed free energy, while the quenched free energy is obtained by minimizing the difference (4.5).

To conclude this section we remark that the quenched entropy $\bar{S}$ and the annealed entropy S_{an} are not necessarily positive for the scales of the entropies depend on the norm chosen for the *a priori* measure μ_0. Indeed changing μ_0 to $C\mu_0$ will change $S(J)$ [(2.6)] to $S(J) + |V| \ln C$. If the normalization of μ_0 is chosen so that $S(J) \geqslant 0$ [e.g., if $g(J,\sigma) \leqslant 1$ for all

$(J, \sigma)]$ then the quenched entropy $\bar{S}$ will be nonnegative; however, the annealed entropy may still be negative, in particular for large values of β. As an example we consider the Ising spin $\frac{1}{2}$ model, i.e.,

$$d\mu_0(\sigma) = \left[\delta(\sigma - 1) + \delta(\sigma + 1) \right] d\sigma \tag{4.6a}$$

For such systems we can always write the interactions ϕ_b in the form

$$\phi_b(\sigma) = \prod_{i \in b} \sigma_i$$

If the probability distributions of the interactions satisfy

$$d\rho_b(x) = d\rho_b(-x) \tag{4.6b}$$

we then obtain

$$F_{\text{an}} = -\frac{1}{\beta}\left[|V| \ln 2 + \sum_{b \subset V} \ln \int e^{\beta x} d\rho_b(x) \right] \tag{4.7}$$

and

$$S_{\text{an}} = |V| \ln 2 + \sum_{b \subset V} \left[\ln \int e^{\beta x} d\rho_b(x) - \beta \frac{\int x e^{\beta x} d\rho_b(x)}{\int e^{\beta x} d\rho_b(x)} \right] \tag{4.8}$$

This shows that S_{an} goes to $|V| \ln 2$ if β goes to zero and ρ_b can be chosen so that S_{an} goes to minus infinity with β going to infinity.[3] Therefore one can imagine that underestimating $\bar{F}$ with an improper choice of f in $\mathscr{B}$ may result that $F[f]$ approaches F_{an} sufficiently closely that a negative entropy will be obtained for large values of β.

5. STABILITY OF THE GAUGE SYMMETRY UNDER BOUNDARY PERTURBATION

The boundary conditions play a predominant role in the definition of infinite volume equilibrium states. In particular if the Hamiltonian has some internal symmetries the interaction with the fixed boundary spins breaks the symmetry and may lead to a "spontaneous breakdown" of this symmetry in the thermodynamic limit.

The random Hamiltonian (2.1) has also internal symmetries. However we shall show in this section that in "pure" models of spin glasses the gauge symmetry cannot be broken by the boundary conditions. This implies that a spin glass cannot generally be characterized by a local order parameter; in fact the order parameter proposed by Edwards and Anderson[(2)] is the expectation value of a nonlocal quantity taken with the measure (2.8) [see (6.2)]. This order parameter can be considered as a local observable only if

[3] For example, $d\rho_b(x) = \rho_b(x)\,dx$ and ρ_b are Gaussian distributions with bounded mean deviations.

one introduces the so-called replicas; in this case, however, one looses the clear description of the quenched states as probability measures. This explains somewhat the difficulty to prove the existence of phase transitions in quenched models.

In this section we consider Hamiltonians of the form

$$H(J,\sigma) = -\sum_{b}\sum_{\alpha=1}^{\nu} J_{b,\alpha}\prod_{i\in b}\sigma_{i,\alpha} = -\sum J_b\sigma_b \tag{5.1}$$

where $\sigma_{i,\alpha}$ is the αth component of the spin at site i, and the $J_{b,\alpha}$'s are independent random variables.

We assume that the interactions have finite range, i.e., there exists some $R > 0$ such that $J_{b,\alpha}$ is strictly zero for any b with $\operatorname{diam}(b) > R$. Furthermore we consider only "pure" models of Spin glasses, i.e., models with even distribution for each interaction:

$$d\rho_{b,\alpha}(x) = d\rho_{b,\alpha}(-x) \tag{5.2a}$$

Finally we assume that the *a priori* measure μ_0 on $S \subset \mathbb{R}^\nu$ is even in all components:

$$d\mu_0(s_1\sigma_1, \ldots, s_\nu\sigma_\nu) = d\mu_0(\sigma_1, \ldots, \sigma_n) \tag{5.2b}$$

$$\forall (s_1, \ldots, s_\nu), \qquad s_i = \pm 1$$

Let $\mathscr{G} = \{s = \{s_{i,\alpha}\}_{1\leqslant\alpha\leqslant\nu}^{i\in L};\ s_{i,\alpha} = \pm 1\}$; for any s in $\mathscr{G}$ we introduce the automorphism τ_s defined on the algebra of local observables by

$$(\tau_s f)(J,\sigma) = f(sJ, s\sigma) \tag{5.3}$$

where

$$(sJ)_{b,\alpha} = J_{b,\alpha}\prod_{i\in b} s_{i,\alpha}$$

$$(s\sigma)_{i,\alpha} = s_{i,\alpha}\sigma_{i,\alpha}$$

These transformations are internal symmetries of the system, since they leave the Hamiltonian and the measure invariant, i.e.,

$$\tau_s H = H$$

and

$$d\mu(s\sigma) = d\mu(\sigma), \qquad d\rho(sJ) = d\rho(J)$$

Let us note that $\mathscr{G}$ is a group of *gauge transformations* since it is generated by transformations involving only a finite number of lattice sites. We shall not consider other symmetries the system might also have.

We are interested in local observables of the form

$$f(J,\sigma) = \prod_{b,\alpha} J_{b,\alpha}^{n_{b,\alpha}} \prod_{i,\beta} \sigma_{i,\beta}^{m_{i,\beta}} \tag{5.4}$$

where $n_{b,\alpha}$ and $m_{i,\beta}$ are nonnegative integers which are different from zero only for a finite number of b's and i's. We introduce the notation

$$\operatorname{supp} f = \bigcup_{b\,:\,n_{b,\alpha}\neq 0} b \bigcup_{i\,:\,m_{i,\beta}\neq 0} \{i\} \subset L$$

For any f of the form 5.4 we have

$$\tau_s f = (-1)^{N_f(s)} f$$

$$N_f(s) = \sum n_{b,\alpha} |V_\alpha(s) \cap b| + \sum m_{i,\beta} |V_\beta(s) \cap \{i\}|$$

$$V_\alpha(s) = \{ j \in L;\ s_{j,\alpha} = -1 \}$$

and therefore f is gauge breaking, i.e., $\tau_s f \neq f$, if $N_f(s_0)$ is odd for some s_0 in $\mathscr{G}$, in which case

$$\tau_s f = -f$$

From the invariance property of H_V and dQ_V we have immediately the following result.

Proposition 5.1. Let f be any gauge-breaking observable of the form (5.4). Then

$$\int f\, dQ_V = 0$$

if $V \supset \operatorname{supp} f$. (A similar result was obtain by Avron *et al.*[6] in the case where f is a pure product of Ising spins.)

To show the absence of spontaneous breakdown of this gauge symmetry we consider quenched states with boundary conditions $dQ_V^{(\text{b.c.})}$. Such states are defined through (2.8) by the Hamiltonian

$$H_V = -\sum_{b \cap V \neq \varnothing} J_b \sigma_b$$

together with the boundary conditions

$$\begin{cases} \sigma_i = \hat{\sigma}_i & \text{if} \quad i \notin V \\ \hat{\rho}_{b,\alpha}(x) & \text{if} \quad b \not\subset V \end{cases} \tag{5.5}$$

For example, for "free" boundary conditions for $\hat{\rho}_{b,\alpha}(x) = \delta(x)$; for "fixed" boundary conditions $\hat{\rho}_{b,\alpha}(x) = \delta(x - \hat{J}_{b,\alpha})$.

Proposition 5.2. Let f be any gauge-breaking local observable of the form (5.4). Then for any boundary conditions (5.5)

$$\langle f \rangle = \int f\, dQ_V^{(\text{b.c.})} = 0$$

if $\operatorname{dist}(\operatorname{supp} f, V^c) > R$.

Proof. Let us take s_0 in $\mathscr{G}$ such that $\tau_{s_0}f = -f$ and $s_i = 1$ for i outside supp f. The measure $dQ_V^{(\mathrm{b.c.})}$ is then invariant under this transformation and yields $\langle f\rangle = -\langle f\rangle = 0$. ■

We notice that this theorem does not exclude the existence of a phase transition in the sense that different weak limits of the quenched measures $dQ_V^{(\mathrm{b.c.})}$ can be obtained, i.e., the possibility still stands for the non-uniqueness of the expectation value of some local gauge-invariant quantity. However, as far as we know, such a hypothesis has never appeared in the literature and can be qualified as "unphysical." An eventual spin glass transition is expected to be characterized by a singularity in the thermodynamic functions and in a nonlocal order parameter. This we discuss in the following section.

6. THE EDWARDS-ANDERSON ORDER PARAMETER

The order parameter q proposed by Edwards and Anderson[2] to describe the spin glass transition is defined as $\langle|\langle\sigma_0\rangle|^2\rangle$ where the first average is the thermal one and the second is taken over the interactions. In order to obtain a nonzero value one has to choose some boundary conditions. We take for the distribution of interactions $J_{b,\alpha}$ across the boundary the same distribution $d\rho_{b,\alpha}$ as the one defining the system and for the external spins some configuration $\hat{\sigma}$; we denote $dQ_V^{\hat{\sigma}}$ the quenched state associated with this boundary condition. With our notation the order parameter q is the thermodynamic limit of

$$\begin{aligned} q_V^{\hat{\sigma}} &= \int d\rho(J) \sum_{\alpha=1}^{\nu} \left[\int d\mu_V(\sigma)\, g_V^{\hat{\sigma}}(J,\sigma)\sigma_{0,\alpha} \right]^2 \\ &= \langle|\langle\sigma_0\rangle_V^{\hat{\sigma}}(J)|^2\rangle \end{aligned} \tag{6.1}$$

provided this limit exists. In (6.1) $g_V^{\hat{\sigma}}(J,\sigma)$ is the probability density (2.4) associated with the boundary condition $\hat{\sigma}$.

We note that $q_V^{\hat{\sigma}}$ has two particular features: Firstly $q_V^{\hat{\sigma}}$ is a nonlocal observable with respect to the quenched measure. Indeed

$$q_V^{\hat{\sigma}} = \int dQ_V^{\hat{\sigma}}\, \sigma_0 \langle\sigma_0\rangle_V^{\hat{\sigma}}(J) \tag{6.2}$$

and $\sigma_0\langle\sigma_0\rangle_V^{\hat{\sigma}}(J)$ is nonlocal since its support is the whole volume occupied by the system.

Secondly $q_V^{\hat{\sigma}}$ is the expectation value of an observable which is invariant with respect to $\mathscr{G}_V = \{s;\ s \in \mathscr{G}\ s_i = 1\ \forall i \notin V\}$. Indeed for any s in $\mathscr{G}$ we have

$$\tau_s\left[\sigma_{0,\alpha}\langle\sigma_{0,\alpha}\rangle_V^{\hat{\sigma}}(J)\right] = \sigma_{0,\alpha}\langle\sigma_{0,\alpha}\rangle_V^{s\hat{\sigma}}(J)$$

therefore $\sigma_{0,\alpha}\langle\sigma_{0,\alpha}\rangle_V^{\hat{\sigma}}(J)$ is invariant under any gauge transformation s in $\mathscr{G}_V$ (see also Avron *et al.*[6]).

Because of this gauge-invariance character q will not distinguish between different gauge-breaking phases. It is, however, an order parameter in the sense that it is zero at high temperatures (Proposition 6.1) and is perhaps nonzero at low temperatures if the dimension of the lattice exceeds some finite value. As we shall see below (Proposition 6.2) q is independent of the boundary conditions $\hat{\sigma}$. This behavior of q implies a new type of low-temperature phase—the spin glass—for the local moment $\langle\langle\sigma_0\rangle\rangle$ vanishes at all temperatures in any model satisfying (5.2) (Proposition 6.3).

Proposition 6.1. Let us consider Ising models ($\sigma_i = \pm 1$) with finite range even interactions[4] and even distribution of spins and bonds (5.2). Suppose the interactions are independent random variables and the number of different distributions is finite. Then the order parameter q vanishes for sufficiently high temperature.

Proof. Using Griffith's inequality[7] and $|\sigma_0| \leqslant 1$ we find that

$$q_V^{\hat{\sigma}} = \langle|\langle\sigma_0\rangle_V^{\hat{\sigma}}(J)|^2\rangle \leqslant \langle\langle\sigma_0\rangle_V^+(|J|)\rangle$$

where $|J| = \{|J_b|\}$ and $+$ means that the boundary spins are positive. Now let R be the range of the interaction and $\lambda = \operatorname{dist}(0, V^c)$. A generalization of Fisher's estimate for pair correlations using self-avoiding walks[8] gives

$$\langle\sigma_0\rangle_V^+(|J|) \leqslant \sum_{n \geqslant \lambda/R} \sum_{\{b_1 \ldots b_n\}}' \tanh\beta|J_{b_1}| \cdots \tanh\beta|J_{b_n}| \tag{6.3}$$

Here the prime indicates that $(b_1 \circ \cdots \circ b_n) \cap V = \{0\}$ and there is no nonempty subset $\{b_{i_1}, \ldots, b_{i_k}\} \subset \{b_1, \ldots, b_n\}$ such that $b_{i_1} \circ \cdots \circ b_{i_k} = \varnothing$ ($A \circ B = A\backslash B \cup B\backslash A$ is the usual symmetric difference of A and B).

Supposing that the lattice is periodic and R is finite, there exists a constant C such that the number of sets $\{b_1, \ldots, b_n\}$ in (6.3) is smaller than C^n. Moreover,

$$0 \leqslant \int_{-\infty}^{\infty} \tanh\beta|x|\, d\rho_b(x) \leqslant \epsilon(\beta) \leqslant 1$$

where $\epsilon(\beta)$ goes to zero with β going to zero. Therefore, for small enough β, $C \cdot \epsilon(\beta) < 1$ and

$$\langle\langle\sigma_0\rangle_V^+(|J|)\rangle \leqslant \sum_{n \geqslant \lambda/R} [C \cdot \epsilon(\beta)]^n \leqslant \frac{[C \cdot \epsilon(\beta)]^{\lambda/R}}{1 - C \cdot \epsilon(\beta)} \tag{6.4}$$

If V increases then λ tends to ∞ and the right-hand side goes to zero. ■

[4] I.e., $J_b \equiv 0$ if $|b|$ is odd.

In the following part of this section we discuss some formulas obtained for Ising models by gauge fixing.

Proposition 6.2. Consider Ising models ($\sigma_i = \pm 1$) with even distribution (5.2) and let $f(J, \sigma)$ be a function of σ_i's for $i \in V$ and of J_b's for $b \cap V \neq \emptyset$. Then (i) if f is invariant under $\mathscr{G}_V$,

$$\int dQ_V^{\hat{\sigma}} f = 2^{|V|} \int d\rho(J)\, g_V^+(J, +) f(\hat{\sigma}J, +) \tag{6.5}$$

where in $\hat{\sigma}J$, $\hat{\sigma}$ is extended to V with the value $\hat{\sigma}_i = 1$ for $i \in V$, and $+$ denotes the configuration $\sigma_i = +1$; (ii) if f is gauge invariant,

$$\int dQ_V^{\hat{\sigma}} f = 2^{|V|} \int d\rho(J)\, g_V^+(J, +) f(J, +) \tag{6.6}$$

Proof. Using the invariance property (5.2) and $\tau_s f = f$ for all $s \in \mathscr{G}_V$ we have

$$\int dQ_V^{\hat{\sigma}} f = \int d\rho(J)\, d\mu(\sigma)\, g_V^{\hat{\sigma}}(J, \sigma) f(J, \sigma) = \int d\rho(J)\, d\mu(\sigma)\, g_V^{\hat{\sigma}}(J, +) f(J, +)$$
$$= \int d\rho(J)\, d\mu(\sigma)\, g_V^+(J, +) f(\hat{\sigma}J, +) = 2^{|V|} \int d\rho(J)\, g_V^+(J, +) f(\hat{\sigma}J, +)$$

which concludes the proof of (i) and (ii). ∎

Let us note that according to (6.6) the expectation value of any gauge-invariant quantity is independent of the boundary condition and can be obtained by "fixing the gauge" at $\sigma_i = +1$ and then averaging with the probability distribution

$$dg_V^+(J) = 2^{|V|} g_V^+(J, +)\, d\rho(J) \tag{6.7}$$

Proposition 6.3. For Ising models ($\sigma_i = \pm 1$) with even distributions (5.2)

$$\begin{aligned} \langle\langle\sigma_A\rangle_V^{\hat{\sigma}} \langle\sigma_B\rangle_V^{\hat{\sigma}}\rangle &= \delta_{A,B} 2^{|V|} \int d\rho(J)\, g_V^+(J, +) \langle\sigma_A\rangle_V^+ \\ &= \delta_{A,B} \langle\langle\sigma_A\rangle_V^+\rangle_{g_V^+} \end{aligned} \tag{6.8}$$

where A and B are subsets of V and σ_A denotes $\prod_{i \in A} \sigma_i$.

Proof. For $A = B$ (6.8) follows immediately from (6.6) since

$$\langle\langle\sigma_A\rangle_V^{\hat{\sigma}} \langle\sigma_B\rangle_V^{\hat{\sigma}}\rangle = \int dQ_V^{\hat{\sigma}} \sigma_A \langle\sigma_B\rangle_V^{\hat{\sigma}}(J) \tag{6.9}$$

$$\tau_s(\sigma_A \langle\sigma_A\rangle_V^{\hat{\sigma}}(J)) = \sigma_A \langle\sigma_A\rangle_V^{s\hat{\sigma}}(J)$$

and

$$\langle\sigma_A\rangle_V^{\hat{\sigma}}(\hat{\sigma}J) = \langle\sigma_A\rangle_V^+(J)$$

For $A \neq B$ we can write

$$\langle\langle\sigma_A\rangle_V^{\hat\sigma}\langle\sigma_B\rangle_V^{\hat\sigma}\rangle = \int d\mu_V(\sigma)\,\sigma_A\sigma_B\left\{\int d\rho(J)\; g_V^{\hat\sigma}(J,\sigma)\sigma_A\langle\sigma_A\rangle_V^{\hat\sigma}(J)\right\}$$

using the fact that

$$\sigma_A\langle\sigma_A\rangle_V^{\hat\sigma}(J) = \langle\sigma_A\rangle_V^{\hat\sigma}(\sigma J)$$

and

$$g_V^{\hat\sigma}(J,\sigma) = g_V^{\hat\sigma}(\sigma J, +)$$

we find

$$\begin{aligned}\langle\langle\sigma_A\rangle_V^{\hat\sigma}\langle\sigma_B\rangle_V^{\hat\sigma}\rangle &= \int d\mu_V(\sigma)\,\sigma_A\sigma_B\left\{\int d\rho(J)\; g_V^{\hat\sigma}(J,+)\langle\sigma_A\rangle_V^{\hat\sigma}(J)\right\}\\ &= 2^{|V|}\delta_{A,B}\left\{\int d\rho(J)\; g_V^{\hat\sigma}(J,+)\langle\sigma_A\rangle_V^{\hat\sigma}(J)\right\}\end{aligned}$$

which concludes the proof. ■

Consequences of Proposition 6.3. (1) For $A = B = \{0\}$ we find

$$q_V^{\hat\sigma} = \langle\langle\sigma_0\rangle_V^+\rangle_{g_V^+} \tag{6.10}$$

which shows that q is independent of the boundary conditions $\hat\sigma$.

(2) For $A = \{i\}$ and $B = \{j\}$ with $i \neq j$ we have

$$\langle\langle\sigma_i\rangle_V^{\hat\sigma}\langle\sigma_j\rangle_V^{\hat\sigma}\rangle = 0 \tag{6.11}$$

This heuristically obvious result was used earlier (see, e.g., Fischer[9]) to conclude that the quenched susceptibility $\bar\chi$ is proportional to $1 - q$; indeed

$$\begin{aligned}\bar\chi_V &\sim \frac{1}{|V|}\sum_{i,j\in V}(\langle\langle\sigma_i\sigma_j\rangle\rangle - \langle\langle\sigma_i\rangle\langle\sigma_j\rangle\rangle)\\ &= \frac{1}{|V|}\sum_{i\in V}(1 - \langle\langle\sigma_i\rangle^2\rangle) + \frac{1}{|V|}\sum_{i\neq j}(\langle\langle\sigma_i\sigma_j\rangle\rangle - \langle\langle\sigma_i\rangle\langle\sigma_j\rangle\rangle)\end{aligned}$$

Using Proposition 6.3 it follows that both terms in the second summation vanish while $\langle\langle\sigma_i\rangle^2\rangle$ tends to q in the thermodynamic limit, at least for a translationally invariant state.

(3) For any A $\langle\langle\sigma_A\rangle\rangle = 0$ independent of the boundary conditions.[6] The inequality

$$\langle(\langle\sigma_A\rangle_V^{\hat\sigma})^2\rangle = \int dg_V^+(J)\,\langle\sigma_A\rangle_V^+(J) \geqslant 0$$

suggests that $dg_V^+(J)$ favors the ferromagnetic interactions. This is correct in the following sense.

Proposition 6.4. For Ising models with even distributions (5.2),

$$\int J_b \, dg_V^+(J) \geqslant 0 \tag{6.12}$$

holds for all interactions.

Proof. It is sufficient to show that $g_V^+(J, +)$ is an increasing function of J_b. Indeed,

$$\begin{aligned}\frac{\partial g_V^+(J,+)}{\partial J_b} &= \frac{\partial}{\partial J_b} \frac{\exp\{\beta \sum_{b' \cap V \neq \varnothing} J_{b'}\}}{\sum_{\sigma_i : i \in V} \exp\{\beta \sum_{b' \cap V \neq \varnothing} J_{b'} \sigma_{b'}\}} \\ &= \beta g_V^+(J,+) - \beta g_V^+(J,+) \langle \sigma_b \rangle_V^+ \geqslant 0 \quad \blacksquare\end{aligned}$$

Corollary. The averaged energy of any bond is nonpositive.

Indeed, the averaged energy of the bond b is

$$-\int J_b \sigma_b \, dQ_V^{\hat{\sigma}} = -\int J_b \, dg_V^+(J) \leqslant 0 \tag{6.13}$$

according to (6.6) and (6.12).

To conclude this section we establish the connection between the order parameter q and the derivative of the quenched free energy, with respect to an external field h. Let $F_V^{\hat{\sigma}}(J)$ be the free energy in volume V with boundary condition $\hat{\sigma}$. Since $F_V^{\hat{\sigma}}(J)$ is gauge invariant, (6.6) implies

$$\int F_V^{\hat{\sigma}} \, dQ_V^{\hat{\sigma}} = 2^{|V|} \int F_V^+(J) g_V^+(J) g_V^+(J,+) \, d\rho(J) \tag{6.14}$$

However,

$$\begin{aligned}\int F_V^{\hat{\sigma}} \, dQ_V^{\hat{\sigma}} &= \int F_V^{\hat{\sigma}}(J) \int g_V^{\hat{\sigma}}(J,\sigma) \, d\mu_V(\sigma) \, d\rho(J) \\ &= \int F_V^{\hat{\sigma}}(J) \, d\rho(J) = \int F_V^+(J) \, d\rho(J) = \bar{F}_V \end{aligned} \tag{6.15}$$

Therefore the quenched free energy is independent of the boundary condition and

$$\bar{F}_V = 2^{|V|} \int F_V^+(J) g_V^+(J,+) \, d\rho(J) \tag{6.16}$$

Let now $F_V^+(J,h)$ be the free energy defined by the equation

$$\exp[-\beta F_V^+(J,h)] = \sum_{\sigma_i : i \in V} \exp\left(\beta \sum_{b \cap V \neq \varnothing} J_b \sigma_b + \beta h \sum_{i \in V} \sigma_i\right)$$

where $\sigma_i = +1$ if $i \in V^c$. Let, moreover,

$$\bar{F}_V(h) = 2^{|V|} \int F_V^+(J,h) g_V^+(J,+) \, d\rho(J) \tag{6.17}$$

We should emphasize that in the above definition g_V^+ does not depend on h. Therefore, $\bar{F}_V(h)$ is not equal to the quenched free energy in the presence of a nonrandom external field, though $\bar{F}_V(0) = \bar{F}_V$. The reason for the introduction of $\bar{F}_V(h)$ through Eq. (6.17) is that it is coupled to the averaged order parameter

$$Q_V = \frac{1}{|V|} \sum_{i \in V} q_V(i) \tag{6.18}$$

where

$$q_V(i) = 2^{|V|} \int d\rho(J)\, g_V^+(J, +) \langle \sigma_i \rangle_V^+(J)$$

just in the same way as in nonrandom models the free energy is coupled to the average magnetization: the comparison of Eqs. (6.18) and (6.17) yields

$$-\frac{\partial}{\partial h}\left(\frac{1}{|V|} \bar{F}_V(h) \right) = Q_V \tag{6.19}$$

Therefore, in a translationally invariant phase the order parameter q can be obtained as the thermodynamic limit of the left-hand side of Eq. (6.19), provided that this limit exists.

7. ANNEALED MODELS WITH ONE, TWO, AND THREE PHASE TRANSITIONS

The annealed models (see Section 2) are usually considered to be trivial and hence of no further interest. This opinion comes from the fact that an annealed model with even distributions for the bonds is in fact equivalent to a model without any interaction [see (4.7) and (7.2)].

Nontrivial results can be obtained either by introducing interactions among the bonds or by destroying the symmetry of their *a priori* distributions. As an example to the former possibility we mention the Ashkin–Teller model[10] in which two consecutive phase transitions were conjectured by Wegner[11] and proved rigorously by Pfister.[12]

Here we exhibit simple examples of annealed models with asymmetric bond distribution, in which one, two, or three phase transitions take place as the temperature changes, depending on the choice of the lattice and of certain parameters.

We shall consider only Ising models, i.e., $\sigma_i = \pm 1$; our discussion is based on the following simple observation:

Let $f = f(\sigma)$ be a local observable; then

$$\langle f \rangle_{\text{an}} = \int dA(J, \sigma)\, f(\sigma) = \int d\mu(\sigma)\, f(\sigma) e^{\beta F_{\text{an}}} \prod_b \int d\rho_b(J_b)\, e^{\beta J_b \sigma_b}$$

But $\sigma_b = \prod_{i \in b} \sigma_i = \pm 1$ implies

$$\langle f \rangle_{\text{an}} = \frac{\int d\mu(\sigma)\, f(\sigma) \exp(\sum_b K_b \sigma_b)}{\int d\mu(\sigma) \exp(\sum_b K_b \sigma_b)} \tag{7.1}$$

where

$$K_b = \frac{1}{2} \ln \left[\frac{\int d\rho_b(x)\, e^{\beta x}}{\int d\rho_b(x)\, e^{-\beta x}} \right] = K_b(\beta) \tag{7.2}$$

Equation (7.1) shows that the annealed system is equivalent to a spin-$\frac{1}{2}$ system with the same lattice and bond structure with fixed interactions $J_b = (1/\beta) K_b(\beta)$. Therefore, the possible phase transitions of the annealed system can be investigated using the known phase transition of the spin-$\frac{1}{2}$ system. It should be stressed that the interactions J_b in the corresponding model are β dependent and this will lead to the existence of several phase transitions. One should also note that

$$K_b(0) = 0$$

and $\operatorname{sign} K_b = \operatorname{sign}[\int d\rho(x) \operatorname{sh} \beta x]$.

In particular, for small β

$$\operatorname{sign} K_b = \operatorname{sign}\left(\int d\rho(x) \cdot x \right)$$

To illustrate the possible existence of several phase transitions we restrict ourselves to the simplest annealed Ising models

$$H = -\sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j \tag{7.3a}$$

with nearest-neighbor interactions distributed according to

$$\rho(J_{ij}) = p\delta(J_{ij} - a) + (1-p)\delta(J_{ij} + b), \qquad a, b > 0 \tag{7.3b}$$

and the *a priori* even measure μ_0

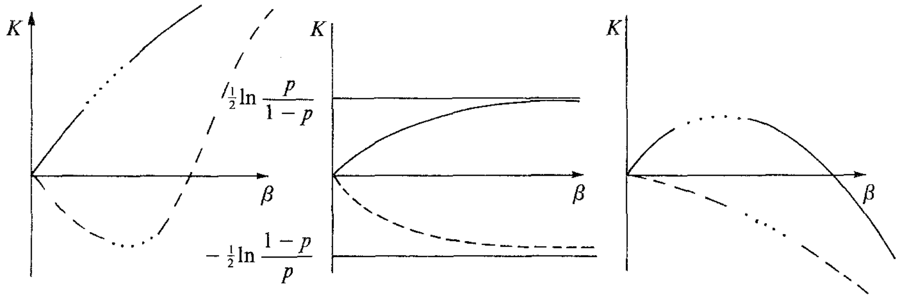
$$\mu_0(\sigma) = \delta(\sigma - 1) + \delta(\sigma + 1) \tag{7.3c}$$

In this case (7.2) yields

$$K_{ij} = K = \frac{1}{2} \ln \left[\frac{p e^{\beta a} + (1-p) e^{-\beta b}}{p e^{-\beta a} + (1-p) e^{\beta b}} \right] \tag{7.4}$$

Therefore for small β $\operatorname{sign} K = \operatorname{sign}[a - ((1-p)/p) b]$

$$\begin{array}{lll} \text{for large } \beta \quad K \sim \frac{1}{2} \beta \cdot \ln(a-b) & \text{if} & a > b \\ \qquad -\frac{1}{2} \beta \cdot \ln(b-a) & \text{if} & a < b \\ \qquad \frac{1}{2} \ln\left(\frac{p}{1-p} \right) & \text{if} & a = b \end{array}$$



-I-	-II-	-III-
(I) $a > b$	(II) $a = b$	(III) $a < b$
(i) —— $a > \frac{1-p}{p}b$	(i) —— $p > \frac{1}{2}$	(i) —— $a > \frac{1-p}{p}b$
(ii) ---- $a < \frac{1-p}{p}b$	(ii) ---- $p < \frac{1}{2}$	(ii) ---- $a < \frac{1-p}{p}b$

In conclusion if the spin-$\frac{1}{2}$ system has an ordered phase for $|K| > K_c$, then the annealed system will have at least one phase transition in the case I and III; it will have at least one phase transition in the case II if $\frac{1}{2}|\ln(p/(1-p))| > K_c$; it will have at least three phase transition in the case I-ii if $K_{\min} < -K_c$ and in the case III-i if $K_{\max} > K_c$.

Proposition 7.1. Let us consider the annealed Ising model (7.3) with $a = b = \bar{J}$ on a d-dimensional simple cubic lattice with $d \geqslant 2$. Let K_c be the critical value of the spin-$\frac{1}{2}$ model with $J_{ij} = J > 0$. Then,

(i) for
$$p > p_c = \frac{e^{2K_c}}{1 + e^{2K_c}} \tag{7.5}$$

there exists two ferromagnetically ordered phases for

$$\beta > \beta_p = \frac{1}{2\bar{J}} \ln\left[\frac{p(1 + e^{2K_c}) - 1}{p(1 + e^{2K_c}) - e^{2K_c}}\right]$$

(ii) for $p < 1 - p_c$ there exists two antiferromagnetically ordered phases for $\beta > \beta_{(1-p)}$.

Let us recall that for $d = 2$ we have $\operatorname{sh} 2K_c = 1$ which yields $p_c = 1/\sqrt{2}$.

Proof. For a given p, $K(\beta)$ given by (7.4) is positive monotonically increasing if $p > \frac{1}{2}$ (resp. negative monotonically decreasing if $p < \frac{1}{2}$) and (7.5) implies that $K(\beta) > K_c$ for $\beta > \beta_p$ [resp. $K(\beta) < -K_c$ for $\beta > \beta_{1-p}$], which concludes the proof. ■

Proposition 7.2. Let us consider the annealed Ising model (7.3) with $b > a > 0$; then there exists some $p_c > \frac{1}{2}$ such that for $p > p_c$:

(i) On the d-dimensional simple cubic lattice with $d \geqslant 2$, there exists $0 < \beta_0 < \beta_1 < \beta_2 < \infty$ such that for $\beta < \beta_0$ and for $\beta_1 < \beta < \beta_2$ there exists a unique (paramagnetic) equilibrium state, for $\beta_0 < \beta < \beta_1$, there exists a ferromagnetic ordering and for $\beta > \beta_2$ there exists an antiferromagnetic ordering.

(ii) On the two-dimensional triangular lattice there exists $0 < \beta_0 < \beta_1 < \infty$ such that for $\beta < \beta_0$ and for $\beta > \beta_1$ there exists a unique (paramagnetic) equilibrium state while for $\beta_0 < \beta < \beta_1$, there exists a ferromagnetic ordering.

Proof. Let $\xi = p/(1-p)$; for fixed β, $K(\xi, \beta)$ Eq. (7.4) is an increasing function of ξ which tends to βa as ξ tends to infinity; therefore for any $a \max_\beta K(\beta/\xi) > K_c$ if $\xi > \xi_c$. Furthermore $\partial K/\partial\beta = 0$ if β is the solution of

$$\operatorname{ch}(a+b)\beta = \frac{a\xi^2 - b}{(b-a)\xi}$$

which is uniquely specified by (a, b, p); therefore for given p, $K(\beta)$ is a concave function which shows that there exists exactly two values β_0 and β_1 such that $K(\beta) = K_c$.

Now on simple cubic lattices the critical temperatures are determined by $K(\beta_0) = K(\beta_1) = K_c$ and $K(\beta_2) = -K_c$. For the triangular lattice $K(\beta_0) = K(\beta_1) = K_c$ and there is some $\tilde{\beta}$ so that $K(\beta) < 0$ for $\beta > \tilde{\beta}$. However, the antiferromagnetic model does not undergo any phase transition at $\beta < \infty$ (Wannier[13]).

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